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iRMB

Compact second-order accurate momentum interpolation for the lattice Boltzmann method in three dimensions

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Outline

Interpolation

Direct versus Compact

Asymptotic analysis

Standard notation versus statistic notation

Grid-refinement

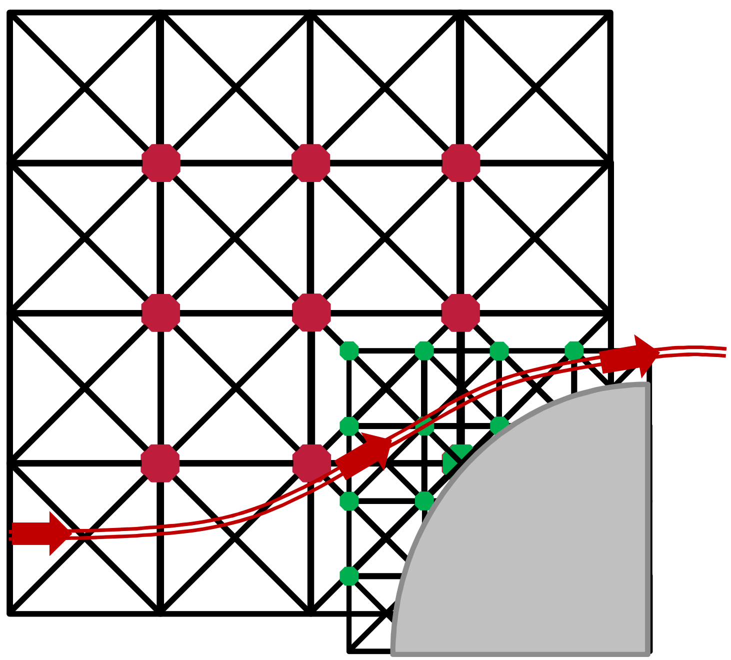
Acoustic versus diffusive scaling

Why interpolation for LBE?

Velocity known at nodes only

Of lattice velocity required in

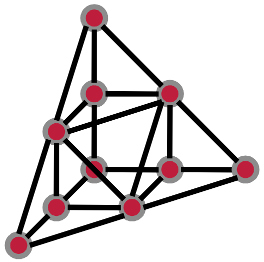
- **Grid-refinement**
- **Curved boundaries**
- **Fluid structure interaction**
- **Tracer/Streamline computation**



How to interpolate?

Required order?

Minimal stencil

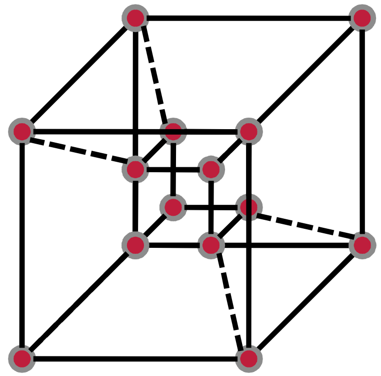


Lattice Boltzmann Equation

- 2^{nd} order accurate for momentum
- 2^{nd} order accurate for shear stress

Consistency with LBE

2^{nd} order interpolation for momentum
1^{st} order interpolation for density



Minimal symmetric stencil

10 nodes stencil

16+ nodes stencil ☹

Is there a more efficient way to gather information for interpolation?

Where to get more information?

Taylor Expansion of Distribution:

$$\sum_{n=0}^{\infty}\frac{\Delta t^n}{n!}\frac{\partial^n}{\partial t^n} f_i = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-\Delta x c_{ix})^k(-\Delta x c_{iy})^m(-\Delta x c_{iz})^l}{k!m!l!}\frac{\partial^k\partial^m\partial^l}{\partial x^k\partial y^m\partial z^l} f_i^*$$

Pre-collision in time

Post-collision in space

Scaling:

Acoustic $\Delta t = \Delta x$

Diffusive $\Delta t = \Delta x^2$

j_x, j_y

p_{xx}, p_{xy}, e

q_x, q_y

ε

"Standard notation"

Expand distribution

Find moments

Derive NSE

μ_x, μ_y

$\mu_{xx}, \mu_{yy}, \mu_{xy}$

μ_{xyy}, μ_{xxy}

μ_{xxyy}

Statistic notation

Why you must NEVER use standard notation

Philosophy of standard notation

Each moment has a given name according to its physical interpretation

j_x : momentum

e : energy

q_x : heat flux

ε : energy square

Not countable

Not extendible

Statistical notation

No physical interpretation

$$\mu_{x^k y^l} = \frac{\sum_i f_i c_{xi}^k c_{yi}^l}{\sum_i f_i}$$

Countable

Extendible to any order

Computations with moments

Can you imagine why Europeans gave up the “standard notation” of numbers in favor for arabic numbers?

CMXLIIX+LX=? ⟷ 948+52=1000

Statistic moment transform inside the expansion

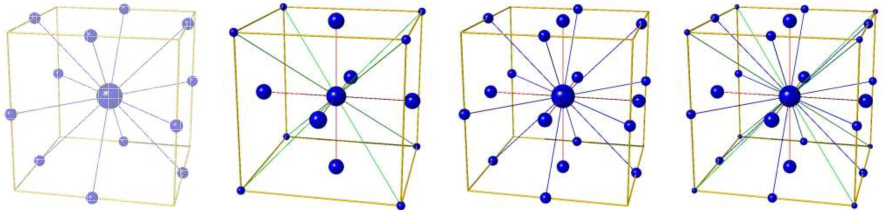
$$\sum_{n=0}^{\infty}\frac{\Delta t^n}{n!}\frac{\partial^n}{\partial t^n}f_i=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\Delta x c_{ix})^k(-\Delta x c_{iy})^m(-\Delta x c_{iz})^l}{k!m!l!}\frac{\partial^k\partial^m\partial^l}{\partial x^k\partial y^m\partial z^l}f_i^*$$

$$\mu_{x^k y^l z^m}=\frac{\sum_i f_i c_{xi}^k c_{yi}^l c_{zi}^m}{\sum_i f_i}$$

$$\sum_{n=0}^{\infty}\frac{\Delta t^n}{n!}\frac{\partial^n}{\partial t^n}\mu_{x^a y^b z^c}=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\Delta x)^{k+m+l}}{k!m!l!}\frac{\partial^k\partial^m\partial^l}{\partial x^k\partial y^m\partial z^l}\mu^*_{x^{a+k}y^{b+m}z^{c+l}}$$

Drastic simplification of asymptotic analysis
Impossible to write down this equation in standard notation

So what information do we get?



$$\mu_x, \mu_y, \mu_z$$

$$\mu_{xx} - \mu_{yy} = \mu_x^2 - \mu_y^2 - \frac{2}{3\omega}\left(\frac{\partial}{\partial x}\mu_x - \frac{\partial}{\partial y}\mu_y\right)$$

$$\mu_{xx} - \mu_{zz} = \mu_x^2 - \mu_z^2 - \frac{2}{3\omega}\left(\frac{\partial}{\partial x}\mu_x - \frac{\partial}{\partial z}\mu_z\right)$$

$$\mu_{xy} = \mu_x\mu_y - \frac{1}{3\omega}\left(\frac{\partial}{\partial x}\mu_y + \frac{\partial}{\partial y}\mu_x\right)$$

$$\mu_{xz} = \mu_x\mu_z - \frac{1}{3\omega}\left(\frac{\partial}{\partial x}\mu_z + \frac{\partial}{\partial z}\mu_x\right)$$

$$\mu_{yz} = \mu_x\mu_y - \frac{1}{3\omega}\left(\frac{\partial}{\partial y}\mu_z + \frac{\partial}{\partial z}\mu_y\right)$$

(D3Q13),D3Q15, D3Q19, D3Q27

Three momenta

Five spatial derivatives of momenta

Eight pieces of information

1st derivatives are in 2nd moments
2nd derivatives in 3rd moments?

What do we get form 3rd moments?

$$\mu_{xxy} = \mu^*_{xxy} - \frac{2}{3}\mu_x\left(\frac{\partial}{\partial x}\mu_y + \frac{\partial}{\partial y}\mu_x\right) - \frac{2}{3}\mu_y\frac{\partial}{\partial x}\mu_x + \frac{2\nu}{3}\left(2\frac{\partial\partial\mu_x}{\partial x\partial y} + \frac{\partial\partial\mu_y}{\partial x\partial x}\right)$$

Known Known Known Known Known Known

2nd derivatives ☺

Viscosity parameter
In general small ☹

Aymptotics versus numerics race

Single precision ↓ Double precision ↓

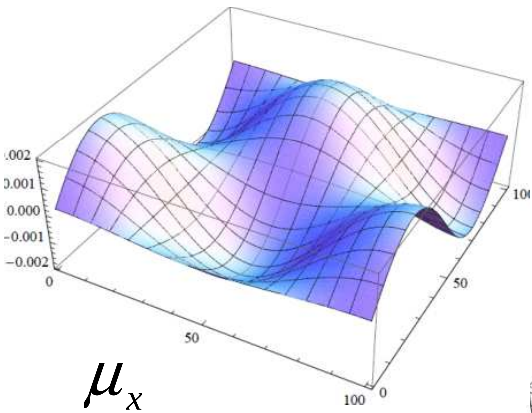
$f =$ 1.11001010101011110101001010100010100101010010100101001000101000111

μ_x ⟷ (Ma) ⟷ $\frac{\partial}{\partial y}\mu_x$ ⟷ (Ma) ⟷ $\frac{\partial\partial}{\partial y\partial y}\mu_x$ ⟷ (ν) ⟷ $\nu\frac{\partial\partial}{\partial y\partial y}\mu_x$

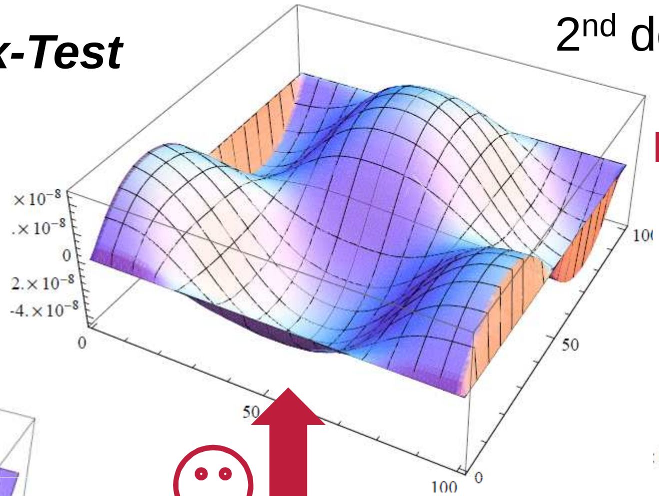
How good are the 2nd derivatives from moments?

Taylor-Green-Vortex-Test

Velocity

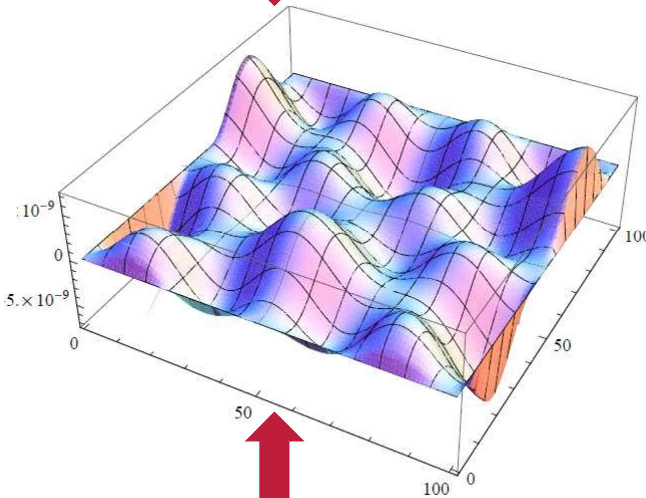


μ_x



2nd derivative from moments

Error 10% ☹



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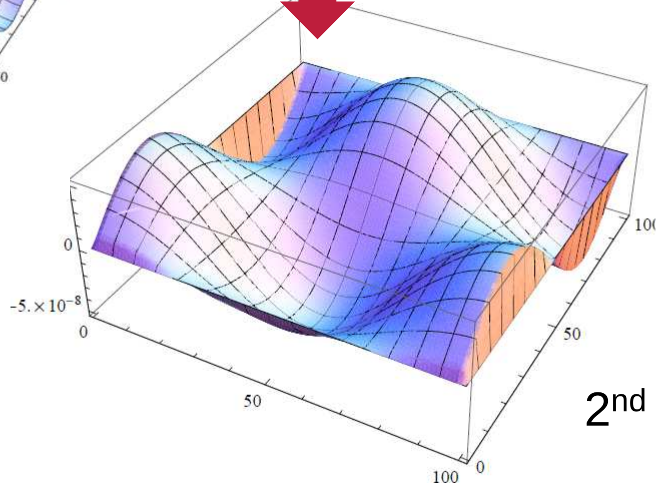


2nd derivative from finite difference

Back to interpolation

With 2nd derivatives not usable we have 8 pieces of information per lattice node

2nd order interpolation formulae have 30 coefficients:

$$\mu_x(x,y,z) = a_0 + a_x x + a_y y + a_z z + a_{xx} x^2 + a_{yy} y^2 + a_{zz} z^2 + a_{xy} xy + a_{xz} xz + a_{yz} yz$$

$$\mu_y(x,y,z) = b_0 + b_x x + b_y y + b_z z + b_{xx} x^2 + b_{yy} y^2 + b_{zz} z^2 + b_{xy} xy + b_{xz} xz + b_{yz} yz$$

$$\mu_x(x,y,z) = c_0 + c_x x + c_y y + c_z z + c_{xx} x^2 + c_{yy} y^2 + c_{zz} z^2 + c_{xy} xy + c_{xz} xz + c_{yz} yz$$

How many nodes required?

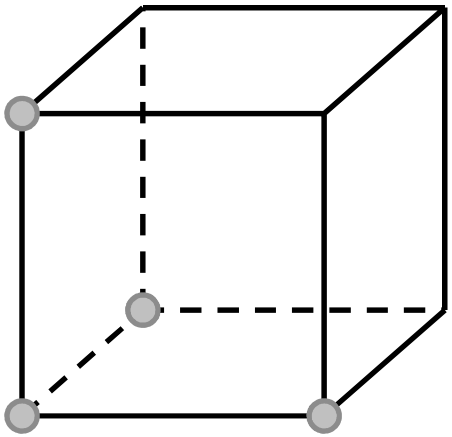
$$\frac{30}{3} = 10$$

10 nodes for usual interpolation

$$\frac{30}{8} = 3.75$$

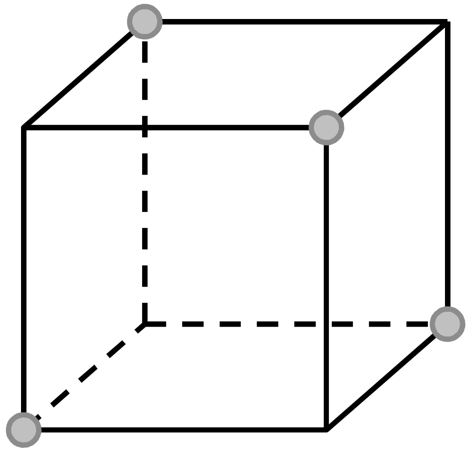
4 nodes for compact interpolation

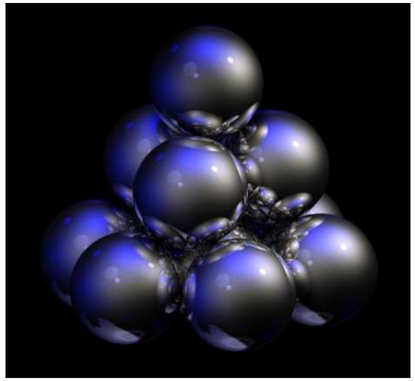
Which nodes would you choose?



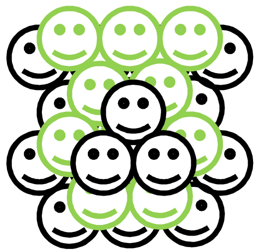
corner

Eight possibilities per cube



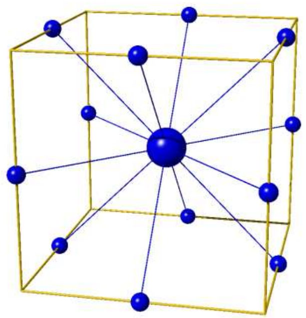


hexagonal close packing
hcp lattice



Two possibilities per cube

D3Q13-ready



D3Q13

But we have too much information

4nodes*8values=32 pieces of information. Two pieces of information too much!

Reduction by:

Least squares
Appears to be optimal but:

μ_x μ_{xy} $\mu_{xx} - \mu_{yy}$

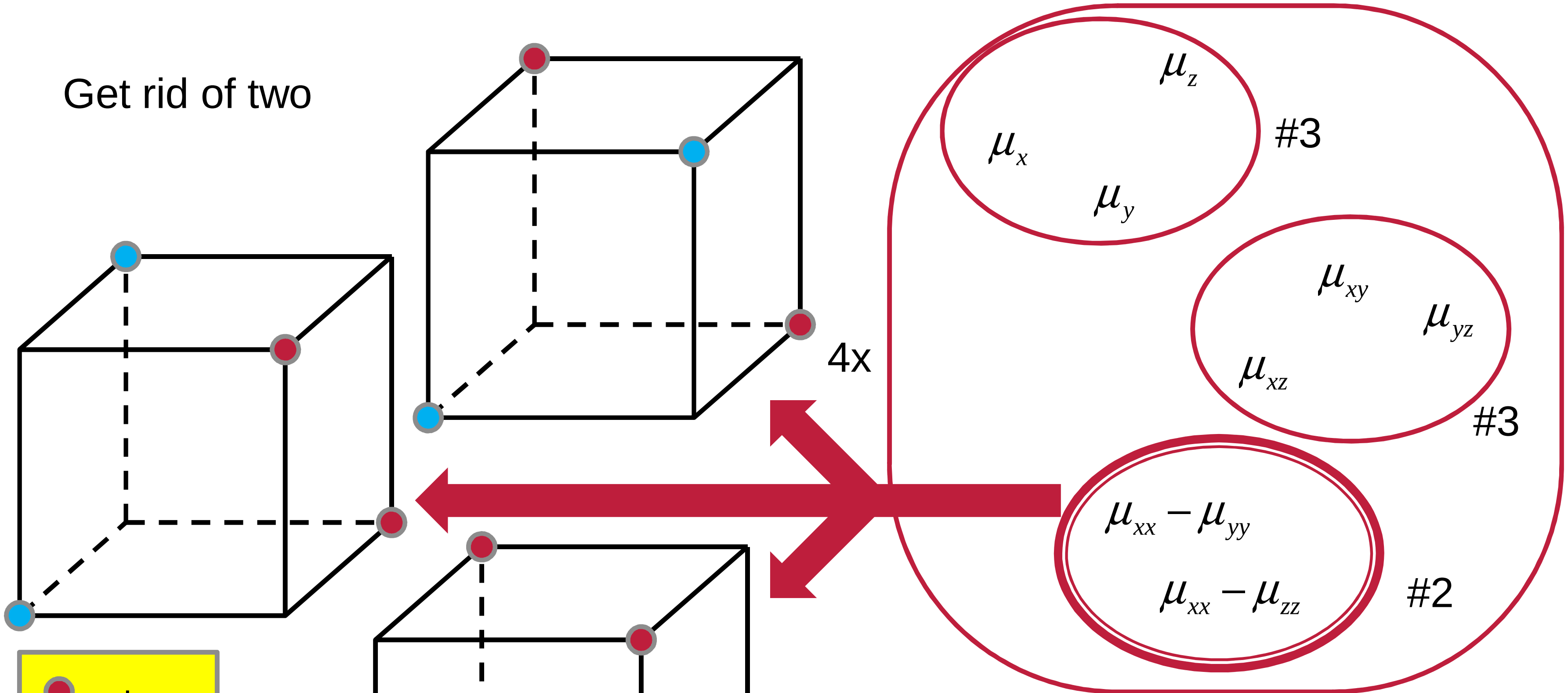
1 2 3 ➡ 3 types of information ➡ Weighting required

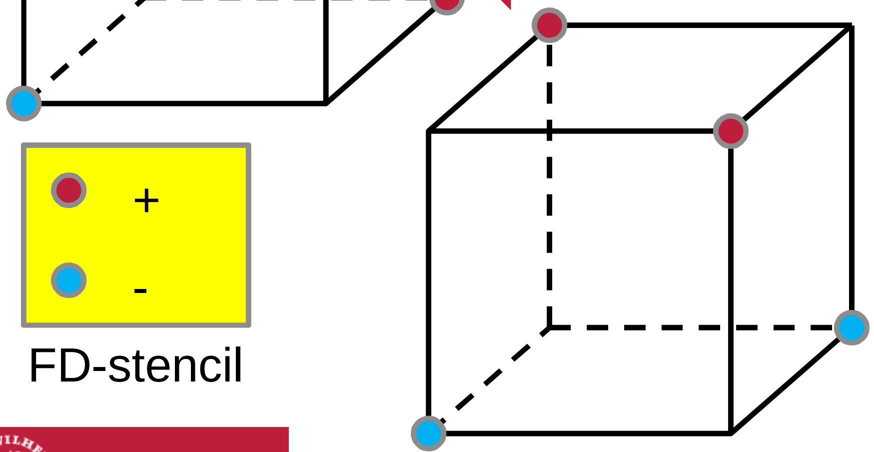
Metric for weighting not known ☹

Asymptotic consistent for any metric but absolute error might vary

Quick & Clean data reduction

Get rid of two



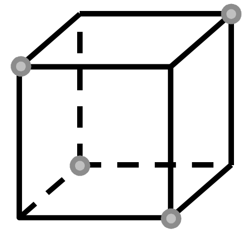


FD-stencil

2x (4 → 3) data reduction by finite differences

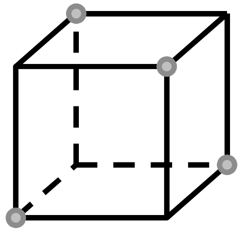
Quick and Clean: FD only 1st order
Coefficients still 2nd order

Application to grid-refinement



D3Q13, D3Q19, D3Q27

- All work with hcp lattice, D3Q19 & D3Q27: two lattices for symmetry



MRT, BGK, CLBE, ELBE

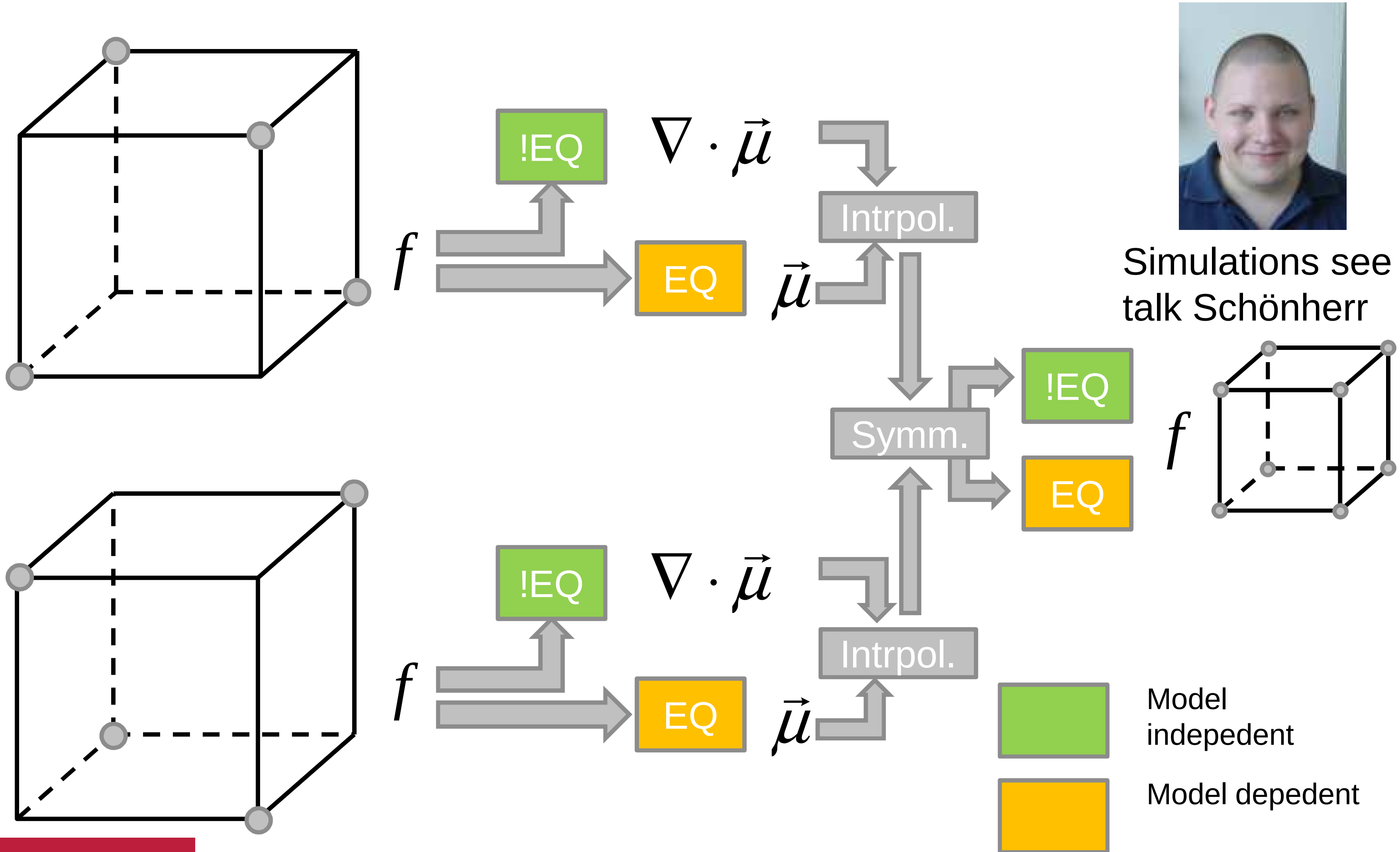
- No differences up to 2nd moments

Compressible formulation, incompressible formulation

- No difference for non-equilibrium, momentum and gradients

  Split off non-equilibrium part

Symmetric compact interpolation for grid-refinement



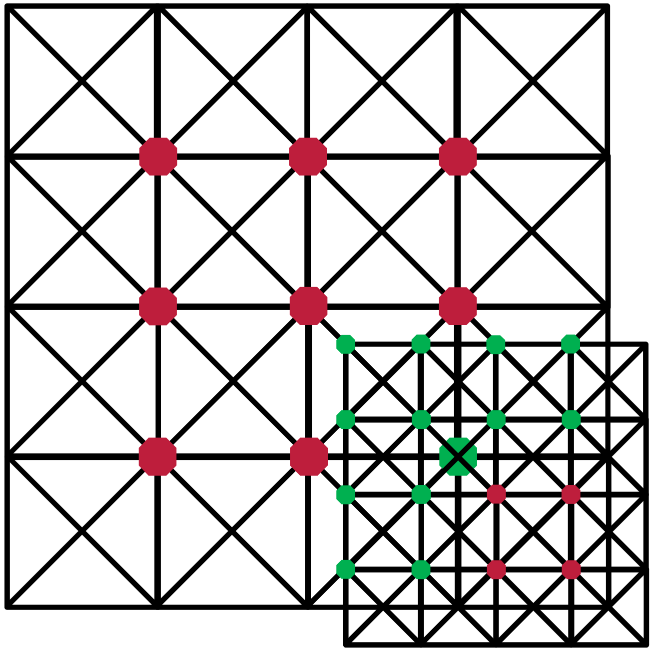
f , $\nabla \cdot \vec{\mu}$, $\vec{\mu}$

Simulations see talk Schönherr

Model indepedent

Model depedent

Time step scaling: acoustic versus diffusive



Δx

Δt

coarse

→

fine

	acoustic	diffusive
	$\frac{\Delta x}{2}$	$\frac{\Delta x}{2}$
	$\frac{\Delta t}{2}$	$\frac{\Delta t}{4}$

Why acoustic scaling in grid coupling?

speed of sound fine=speed of sound coarse
➡ ***Zero reflection condition***

Reduce asymptotics versus numerics race

Efficiency?

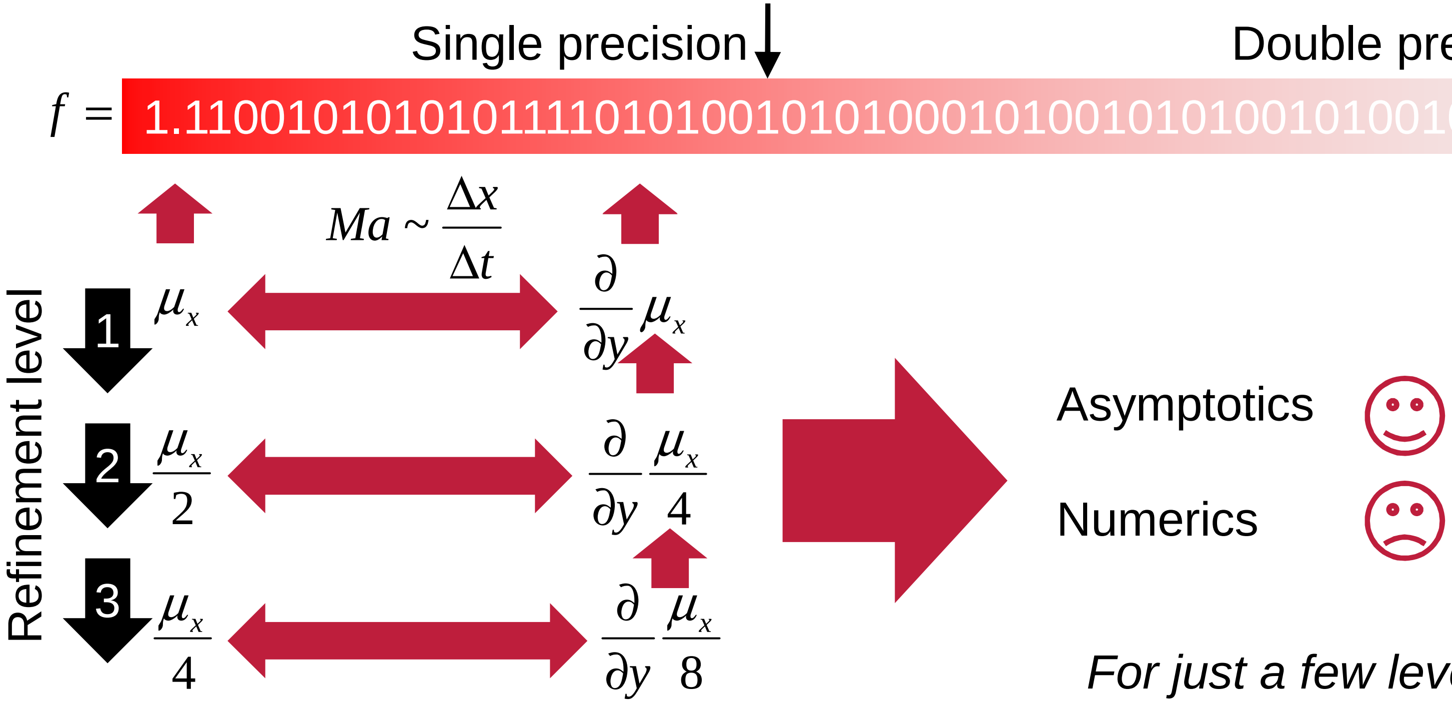
Refinement ***less*** *expensive for acoustic scaling*

Coarsening ***more*** *expensive for acoustic scaling*

Asymptotics versus numerics race

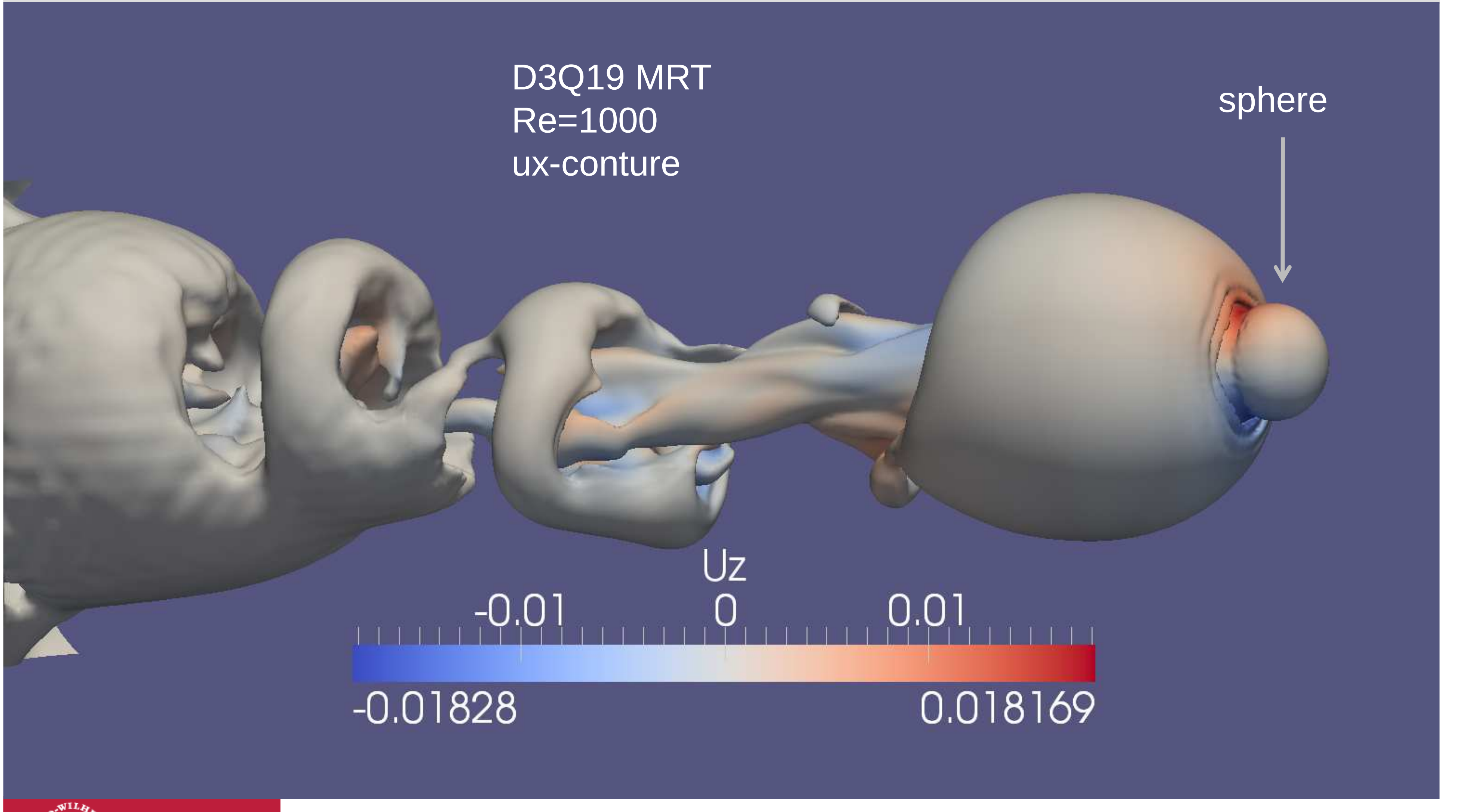
Acoustic scaling: $\frac{\Delta x}{\Delta t} = const$ on all levels

Diffusive scaling: one bit of significance lost per grid-level



For just a few levels
this might not matter…

Grid-refinement 3D



Summary

Do not use standard notation for moments!

→ Use statistic notation

Asymptotic expansion

→ Best done directly in moment space

Compact interpolation

→ Small stencil
Asymptotic consistent
Symmetry at lower cost

Sponsors

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Thank you!